Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00023-0

J. Appl. Maths Mechs, Vol. 66, No. 2, pp. 179–186, 2002

0021-8928/02/$—see front matter

THE STABILITY OF STEADY MOTIONS OF NON-HOLONOMIC CHAPLYGIN SYSTEMS†

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(*Received 2 April 2001*)

A stability theorem is established for steady motions of non-holonomic Chaplygin systems, with cyclic coordinates, acted upon by potential and dissipative forces, generalizing a previously proved theorem [1]. The theorem enables rigorous sufficient conditions for the stability of steady motions of non-holonomic systems to be derived in cases that are more general than those considered hitherto. As an example, the problem of the stability of the steady motions of a one-wheeled carriage is considered. © 2002 Elsevier Science Ltd. All rights reserved.

For a survey of results on the theory of the stability of the steady motions of non-holonomic systems see [2, 3].

1. STEADY MOTIONS

Consider a non-holonomic mechanical system whose state is defined by generalized coordinates $q_1, \ldots, q_n$. The velocities $\dot{q}_1, \ldots, \dot{q}_n$ are subject to $n - l$ $(l < n)$ time-independent non-holonomic constraints

$$\dot{q}_\chi = \sum_{r=1}^{l} b_{\chi r}(q)\dot{q}_r, \quad \chi = l+1, \ldots, n \tag{1.1}$$

It is assumed that the system is acted upon by potential forces, which are the derivatives of a force function U, and by dissipative forces, which are the derivatives of a Rayleigh function F. We will assume that the kinetic energy T, the force function U, the function F and the coefficients $b_{\chi r}$ are independent of q_χ.

The equations of motion of the system in Chaplygin form are, as is well known [1–4]

$$\frac{d}{dt}\frac{\partial \theta}{\partial \dot{q}_r} - \frac{\partial \theta}{\partial q_r} - \sum_{\chi=l+1}^{n} \sum_{s,p=1}^{l} \theta_{\chi p} \nu_{\chi rs} \dot{q}_s \dot{q}_p = \frac{\partial U}{\partial q_r} - \frac{\partial \Phi}{\partial \dot{q}_r}, \quad r = 1, \ldots, l \tag{1.2}$$

where θ, θ_χ and Φ are the results of eliminating the velocities $\dot{q}_\chi$, by using (1.1), from the expressions for T, $\partial T/\partial \dot{q}_\chi$ and F, that is

$$2\theta = \sum_{r,s=1}^{l} a_{rs}(q)\dot{q}_r\dot{q}_s, \quad \theta_\chi = \sum_{s=1}^{l} \theta_{\chi s}\dot{q}_s, \quad 2\Phi = \sum_{r,s=1}^{l} f_{rs}(q)\dot{q}_r\dot{q}_s, \quad \nu_{\chi rs} = \frac{\partial b_{\chi r}}{\partial q_s} - \frac{\partial b_{\chi s}}{\partial q_r}$$

Equations (1.2) constitute a closed system in the variables $q_1, \ldots, q_l$, which can be investigated independently of the equations of the non-holonomic constraints (1.1).

We will assume that among the coordinates $q_1, \ldots, q_l$ there are cyclic coordinates q_α $(\alpha = k + 1, \ldots, l)$ in the sense of the definition in [1, 2], that is

$$\frac{\partial \theta}{\partial q_\alpha} = 0, \quad \frac{\partial}{\partial q_\alpha} \sum_{\chi=l+1}^{n} \theta_{\chi p} \nu_{\chi rs} = 0, \quad \frac{\partial U}{\partial q_\alpha} = 0, \quad \frac{\partial \Phi}{\partial q_\alpha} = 0; \quad p, s, r = 1, \ldots, l$$

The other coordinates q_i $(i = 1, \ldots, k)$ are positional coordinates.

†*Prikl. Mat. Mekh.* Vol. 66, No. 2, pp. 192–199, 2002.

In the context of this definition of cyclic coordinates, steady motions may exist, in which the positional coordinates and the cyclic velocities are constant

$$q_i(t) = q_{i0}, \quad \dot{q}_i(t) = 0, \quad i = 1, \dots, k; \quad \dot{q}_\alpha(t) = \dot{q}_{\alpha 0} = \omega_\alpha, \quad \alpha = k+1, \dots, l \tag{1.3}$$

A necessary condition for the existence of steady motions (1.3) is that there must be no dissipation with respect to the cyclic velocities, that is

$$\partial \Phi / \partial \dot{q}_\alpha = 0, \quad \alpha = k+1, \dots, l$$

In that case the l constants q_{i0} and ω_α satisfy the l equations

$$\left(\frac{\partial U}{\partial q_i} \right)_0 + \sum_{\alpha,\beta=k+1}^{l} \left(\frac{1}{2} \frac{\partial a_{\alpha\beta}}{\partial q_i} + \sum_{\chi=l+1}^{n} \theta_{\chi\beta} \nu_{\chi i \alpha} \right)_0 \omega_\alpha \omega_\beta = 0, \quad i = 1, \dots, k \tag{1.4}$$

$$\sum_{\alpha,\beta=k+1}^{l} \left(\sum_{\chi=l+1}^{n} \theta_{\chi\alpha} \nu_{\chi\gamma\beta} \right)_0 \omega_\alpha \omega_\beta = 0, \quad \gamma = k+1, \dots, l \tag{1.5}$$

The subscript zero means that the expression is evaluated for steady motion.

It has been observed [2, 3] that system (1.4), (1.5) generally has only trivial solutions

$$\omega_\alpha = 0, \quad q_i(t) = q_{i0} \quad (q_{i0} : \partial U / \partial q_i = 0)$$

corresponding to equilibrium positions of the system. In some cases, however, it may happen that only l_1 ($l_1 < l$) of Eqs (1.4) and (1.5) are independent, in which case system (1.4), (1.5) may have non-trivial solutions $\dot{q}_\alpha$. The mechanical system under consideration may then have an $(l - l_1)$-dimensional family of steady solutions (1.3).

If the condition

$$\sum_{\chi=l+1}^{n} (\theta_{\chi\alpha} \nu_{\chi\gamma\beta})_0 = -\sum_{\chi=l+1}^{n} (\theta_{\chi\beta} \nu_{\chi\gamma\alpha})_0, \quad \gamma = k+1, \dots, l \tag{1.6}$$

is satisfied, then condition (1.5) is satisfied for any ω_α. Then the system admits of a manifold of steady solutions of dimension no less than the number of cyclic coordinates $l - k$. Condition (1.6) is satisfied, in particular, if

$$\sum_{\chi=l+1}^{n} (\theta_{\chi\alpha} \nu_{\chi\gamma\beta})_0 = 0, \quad \alpha, \beta, \gamma = k+1, \dots, l \tag{1.7}$$

Obviously, a sufficient condition for (1.7) to hold is that [1–3]

$$\sum_{\chi=l+1}^{n} \theta_{\chi\alpha} \nu_{\chi\gamma\beta} = 0, \quad \alpha, \beta, \gamma = k+1, \dots, l \tag{1.8}$$

Note that conditions (1.8) hold identically with respect to the positional coordinates, but conditions (1.7) hold only in steady motion.

This last situation (conditions (1.8) hold) is precisely that arising in the well-known problems of the steady motions of a heavy rigid body (a disk, torus, etc.) on an absolutely rough horizontal plane. In the example presented below, conditions (1.8) will fail to hold, but conditions (1.7) will be satisfied.

2. STABILITY ANALYSIS

Consider an arbitrary point of the manifold of steady motions (1.4), (1.5) and let us investigate the problem of whether solution (1.3) is stable with respect to perturbations of the variables q_i, $\dot{q}_i$ and $\dot{q}_\alpha$.

We introduce the deviations

$$x_i = q_i - q_{i0}, \quad y_\alpha = \dot{q}_\alpha - \omega_\alpha, \quad i = 1, \dots, k; \quad \alpha = k+1, \dots, l$$

and write down the equations of perturbed motion in matrix form, with the linear terms isolated

$$\begin{aligned} A\ddot{x} + C\dot{y} &= W_1 x + D_1\dot{x} + P_1 y + X(x, \dot{x}, y) \\ C^T\ddot{x} + B\dot{y} &= W_2 x + D_2\dot{x} + P_2 y + Y(x, \dot{x}, y) \end{aligned} \tag{2.1}$$

where $x(k \times 1)$, $y(s \times 1)$; $s = l - k$. The formulae for the elements of the matrices A, B, ... are similar to the corresponding formulae in [2]; X and Y are vector-valued functions containing terms of order greater than one in the variables just introduced.

When condition (1.8) is satisfied, $W_2 = 0$ and $P_2 = 0$. When conditions (1.6) or (1.7) are satisfied, only $P_2 = 0$.

It is obvious that if all the roots of the characteristic equation of the linearized system (2.1) are in the left half-plane, then the trivial solution of system (2.1) is asymptotically stable, but if at least one root is in the right-half-plane, it is unstable.

If $W_2 = 0$ and $P_2 = 0$, system (2.1) has s zero roots and s linear integrals. It has been shown [1, 2] that this situation corresponds to the special critical case, in Lyapunov's sense, of several zero roots. Under the conditions specified above, the Lyapunov–Malkin theorem [5, 6] has been used to established a stability theorem for the steady motion (1.3) [1].

We shall show that, under certain conditions, a similar theorem will hold for Chaplygin systems of more general form, possessing as s-dimensional manifold of steady motions. In that case the linearized system will also have s zero roots and s linear integrals (one s-dimensional vector integral), but on the other hand $W_2 \neq 0$ and $P_2 \neq 0$.

Let us find the conditions under which such a vector linear integral in the linearized system (2.1) exists. Introducing matrices $L(s \times k)$ and $M(s \times s)$, one readily sees that, if a non-trivial solution L, M of the system of matrix equations

$$LW_1 + MW_2 = 0, \qquad LP_1 + MP_2 = 0 \tag{2.2}$$

exists, then the linearized system (2.1) has an $(s \times 1)$-dimensional vector linear integral, of the form

$$z = (LA + MC^T)\dot{x} + (LC + MB)y - (LD_1 + MD_2)x = \text{const} \tag{2.3}$$

System (2.2) has a non-zero solution if

$$\Delta = \det\begin{Vmatrix} W_1 & P_1 \\ W_2 & P_2 \end{Vmatrix} = 0 \tag{2.4}$$

This condition is satisfied, in particular, if $W_2 = 0$ and $P_2 = 0$ (see [1, 2]).

Let $\det W_1 \neq 0$. Then it follows from system (2.2) that $L = -MW_2W_1^{-1}$, $MP_0 = 0$, where $P_0 = P_2 - W_2W_1^{-1}P_1$. We have $M \neq 0$ if $\det P_0 = 0$ ($0 \leqslant \operatorname{rank} P_0 \leqslant s$).

A necessary condition for eliminating the variable y using integral (2.3) is that $\operatorname{rank} G = s$, where

$$G = LC + MB = MB_0, \qquad B_0 = B - W_2W_1^{-1}C$$

and moreover $\operatorname{rank} G \leqslant \min\{\operatorname{rank} M, \operatorname{rank} B_0\}$. Since $\operatorname{rank} M \leqslant s$, the matrix M must be non-singular, and it follows from the relation $MP_0 = 0$ that $P_0 = 0$, that is

$$P_2 = W_2W_1^{-1}P_1 \tag{2.5}$$

Then $\operatorname{rank} G = \operatorname{rank} B_0 = s$ and the condition for y to be eliminated from integral (2.3) is that

$$\det B_0 = \det(B - W_2W_1^{-1}C) \neq 0 \tag{2.6}$$

As the matrix M we can take the identity matrix, and $L = -W_2W_1^{-1}$; $G = B_0$.

Making the change of variables

$$y = B_0^{-1}(z + D_0x - C_0^T\dot{x}) \tag{2.7}$$

we can reduce Eqs (2.1) to the form

$$A_0\ddot{x} + D_0\dot{x} + W_0 x - P_1 B_0^{-1} z = X_0(x, \dot{x}, z), \quad \dot{z} = Z_0(x, \dot{x}, z) \tag{2.8}$$

where

$$A_0 = A - CB_0^{-1}C_0^T, \quad D_0 = -D_1 + P_1 B_0^{-1} C_0^T + CB_0^{-1} D_3$$

$$W_0 = -W_1 - P_1 B_0^{-1} D_3, \quad C_0^T = C^T - W_2 W_1^{-1} A, \quad D_3 = D_2 - W_2 W_1^{-1} D_1$$

The functions $X_0(x, \dot{x}, z)$ and $Y_0(x, \dot{x}, z)$ are formed from the functions $X(x, \dot{x}, z)$ and $Y(x, \dot{x}, z)$ using the change of variables (2.7).

Obviously, the characteristic equation of the linearized system (2.8) will always have s zero roots, while the remaining roots will satisfy the equation

$$\det(A_0\lambda^2 + D_0\lambda + W_0) = 0 \tag{2.9}$$

If one of the roots of Eq. (2.9) has a positive real part, steady motion (1.3) is unstable by Lyapunov's theorem on instability in the first approximation. Since under the conditions specified above the number of zero roots is precisely the number of dimensions of the manifold of steady motions (1.3) (just as in the case considered in [1]), it follows that, if all the roots of Eq. (2.9) have negative real parts, this is the special critical case of several zero roots and the Lyapunov–Malkin theorem holds.

We have thus established a proposition similar to a theorem formulated in [1].

Theorem. A steady motion (1.3) of a non-holonomic Chaplygin system, possessing a manifold of steady motions of dimensionality equal to the number of cyclic coordinates, is stable (unstable) if all the roots of Eq. (2.9) have negative real parts (there is at least one root with positive real part). In the stable case, any perturbed motion sufficiently close to the unperturbed motion will tend to one of the possible steady motions in the aforementioned manifold as $t \to \infty$.

3. THE STEADY MOTIONS OF A MONOCYCLE

Consider a mechanical system modelling a one-wheeled controllable carriage moving on a fixed horizontal plane [7, 8]. The system consists of a homogeneous circular disk of mass m_1 and radius b, rolling without slipping on a plane, of a rigid body M_2 attached at the centre of the disk O_1 by a cylindrical hinge and moving in the plane of the disk; the body axis $O_1\eta'$ lies in the plane of the disk and is a principal axis of inertia of the body M_2 for the point O_1. Let O_2, the centre of mass of the body M_2, lie on the axis $O_2\eta'(O_1O_2 = d)$. Mounted on this same axis is a homogeneous symmetrical flywheel, whose centre of mass coincides with O_2. Let m_2 denote the mass of the body M_2 together with the flywheel.

A simpler model of a monocycle, consisting of a disk, a weightless rod and a ball revolving around it, was considered in [9].

To describe the motion of the system, let us introduce a fixed system of coordinates $OXYZ$ with origin at some point of the support plane (the Z axis points vertically upwards) and a half-attached system of coordinates $O_1\xi\eta\zeta$ with origin at the centre of mass O_1 of the disk: the ζ axis is perpendicular to the plane of the disk and the η axis points along the straight line of maximum inclination of the plane of the wheel (upwards). The position of the system is defined by generalized coordinates X, Y, θ, ψ, φ, φ_1, α, where X and Y are the horizontal coordinates of the centre of mass of the disk, θ, ψ and φ are the Euler angles defining the position of the disk, φ_1 is the angle between the axes $O_1\eta$ and $O_1\eta'$, which characterizes the position of the $O_1\eta'$ axis of the body M_2, the third coordinate of the centre of mass of the disk is $Z = b\cos\theta$ and α is the angle of rotation of the flywheel (rotor) relative to the body M_2. The vectors $\dot{\alpha}$ and $\dot{\varphi}$ point along the $O_1\eta'$ and $O_1\zeta$ axes, respectively.

The equations of the non-holonomic constraints express the no-slip conditions (the velocity of the point P of the disk at which it is in contact with plane is zero) and have the form

$$\begin{aligned} &\dot{X} + b[\dot{\theta}\sin\psi\cos\theta + (\dot{\varphi} + \dot{\psi}\sin\theta)\cos\psi] = 0 \\ &\dot{Y} + b[-\dot{\theta}\cos\psi\cos\theta + (\dot{\varphi} + \dot{\psi}\sin\theta)\sin\psi] = 0 \end{aligned} \tag{3.1}$$

The expressions for the kinetic energy of the carriage and the force function in generalized coordinates were presented in [7].

We will assume that the system is subject to dissipative forces which are the derivatives of the Rayleigh function $F = h_1\dot{\varphi}_1^2/2$ (h_1 = const > 0), that model the forces of viscous friction in the wheel axis.

The kinetic energy and force function of the system, as well as the coefficients of the non-holonomic constraints (3.1), are independent of the X and Y coordinates; hence the system is a Chaplygin system and its equations of motion, set up in the form of Chaplygin's equations, may be investigated independently of the constraint equations (3.1). The coordinates φ, ψ and α are cyclic in the sense of the definition in [2].

The equations of motion admit of particular solutions

$$\varphi_1 = \varphi_{10}, \quad \dot{\varphi}_1 = 0, \quad \theta = \theta_0, \quad \dot{\theta} = 0, \quad \dot{\varphi} = \dot{\varphi}_0 = \omega, \quad \dot{\psi} = \dot{\psi}_0 = \Omega, \quad \dot{\alpha} = \Omega_r \tag{3.2}$$

that describe steady motions of the system.

The conditions for steady motion (3.2) to exist are

$$\begin{aligned}&\{I_2 \sin\theta_0 \omega\Omega + [I_2 \sin^2\theta_0 + (B_2 - A_2)\cos^2\theta_0 \cos\varphi_{10}]\Omega^2 + \\ &+ I\Omega\Omega_r \cos\theta_0 - m_2 dg \cos\theta_0\} \sin\varphi_{10} = 0\end{aligned} \tag{3.3}$$

$$\begin{aligned}&C_0\Omega\omega\cos\theta_0 + [(m_1 + m_2)a + m_2 d\cos\varphi_{10}]g\sin\theta_0 - I\Omega_r\Omega\sin\theta_0\cos\varphi_{10} + \\ &+ [C_0 + C_2 - A - B_2\cos^2\varphi_{10} - A_2\sin^2\varphi_{10} + I_2\cos\varphi_{10}]\Omega^2\sin\theta_0\cos\theta_0 = 0\end{aligned} \tag{3.4}$$

$$I_2 \sin\varphi_{10}\Omega^2 = 0, \qquad I_2 \sin\varphi_{10}\Omega\omega = 0 \tag{3.5}$$

where

$$A_2 = A_1 + m_2 d^2, \quad B_2 = B_1, \quad C_2 = C_1 + m_2 d^2, \quad C_0 = C + (m_1 + m_2)b^2 + I_2 \cos\varphi_{10}, \quad I_2 = m_2 bd$$

$A = B$ and C are the principal central moments of inertia of the disk about the diameter and an axis perpendicular to the plane of the disk, B_1, A_1 and C_1 are the principal central moments of inertia of the body M_2 (together with the attached rotor) about the axes $O_2\eta'$, $O_2\xi'$, $O_2\zeta'$ (the $O_2\zeta'$ axis is perpendicular to the plane of the disk and the $O_2\xi'$ axis lies in the plane of the disk), I is the moment of inertia of the rotor about the axis of rotation.

Equations (3.5) correspond to the group of conditions (1.5). It follows from Eqs (3.3)–(3.5) that a necessary condition for the existence of steady motions is that $\sin\varphi_{10} = 0$, which indeed guarantees satisfaction of condition (1.7) and thereby also of condition (1.5) for any values of the cyclic velocities. It is important to observe that condition (1.8) does not hold in this problem.

Thus, in this problem a manifold of steady motions of dimension $s = 3$ exists, and moreover the parameters θ_0, ω, Ω and Ω_r are related by (3.4), in which $\sin\varphi_{10} = 0$.

The condition $\sin\varphi_{10} = 0$ shows that in any steady motion the axis of the body M_2 (the axis of the main body of the carriage) must coincide with the straight line of maximum inclination of the wheel. When that is the case, the centre of mass O_2 of the body M_2 lies above the centre of the wheel O_1 if $\varphi_{10} = 0$ ($\varepsilon = 1$) and below it if $\varphi_{10} = \pi$ ($\varepsilon = -1$).

A few of the steady motions deserve special mention.

1. $\theta_0 = 0$, $\dot{\varphi} = \omega \neq 0$, $\dot{\psi} = \Omega = 0$, $\dot{\alpha} = \Omega_r$ – rolling of a vertical disk in a straight line, with the centre of mass O_1 of the disk moving at an arbitrary constant velocity $|\omega a|$; the point O_2, as before, lies on the vertical diameter of the disk.

2. $\theta_0 = 0$, $\dot{\varphi} = \omega = 0$, $\dot{\psi} = \Omega \neq 0$, $\dot{\alpha} = \Omega_r$ – rotation of the disk at an arbitrary constant angular velocity Ω about a fixed vertical diameter (spinning); the centre of mass O_2 is also on the vertical diameter.

In the general case, when

$$\theta_0 \neq 0, \quad |\theta_0| < \pi/2, \quad \dot{\varphi} = \omega, \quad \dot{\psi} = \Omega \neq 0, \quad \dot{\alpha} = \Omega_r$$

steady motion of the system is motion of the disk at a constant angle of inclination to the support plane; the centre of mass O_1 of the disk and the point of contact P with the plane move around circles of fixed radius; the centre of mass of the body M_2 is on the straight line of maximum inclination of the wheel.

The equations of perturbed motion, linearized in the neighbourhood of the steady motion (3.2), have the form corresponding to (2.1) [7]

$$W_{11}\ddot{x}+W_{12}\dot{y}+V_{11}\dot{x}+K_{11}x+P_1 y=0,\quad W_{21}\ddot{x}+W_{22}\dot{y}+V_{21}\dot{x}+K_{21}x=0 \tag{3.6}$$

where

$$x=(x_1x_2)^T,\qquad y=(y_1y_2y_3)^T$$

$$x_1=\varphi_1-\varphi_{10},\quad x_2=\theta-\theta_0,\quad y_1=\dot{\varphi}-\omega,\quad y_2=\dot{\psi}-\Omega,\quad y_3=\dot{\alpha}-\Omega_r$$

$$\left\|\begin{matrix} \underset{(2\times2)}{W_{11}} & \underset{(2\times3)}{W_{12}} \\ \underset{(3\times2)}{W_{21}} & \underset{(3\times3)}{W_{22}} \end{matrix}\right\| = \left\|\begin{matrix} w_{11} & 0 & w_{13} & w_{14} & 0 \\ 0 & w_{22} & 0 & 0 & 0 \\ w_{13} & 0 & w_{33} & w_{34} & 0 \\ w_{14} & 0 & w_{34} & w_{44} & w_{45} \\ 0 & 0 & 0 & w_{54} & w_{55} \end{matrix}\right\|$$

$$\left\|\begin{matrix} \underset{(2\times2)}{V_{11}} \\ \underset{(3\times2)}{V_{21}} \end{matrix}\right\| = \left\|\begin{matrix} h_1 & v_{12} \\ -v_{12} & 0 \\ 0 & v_{32} \\ 0 & v_{42} \\ 0 & v_{52} \end{matrix}\right\|,\quad \left\|\begin{matrix} \underset{(2\times2)}{K_{11}} \\ \underset{(3\times2)}{K_{21}} \end{matrix}\right\| = \left\|\begin{matrix} k_{11} & 0 \\ 0 & k_{22} \\ k_{31} & 0 \\ k_{41} & 0 \\ 0 & 0 \end{matrix}\right\|,\quad P_1=\left\|\begin{matrix} 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} \end{matrix}\right\|$$

The expressions for the matrix coefficients may be found in [7].

The conditions (2.5) and (2.6) for linear integrals to exist are satisfied, because

$$P_2=0,\quad W_2W_1^{-1}P_1=0$$

Note that for an arbitrary steady motion (3.2) we have $W_2\neq 0$, but in rectilinear motion $W_2=0$ and $P_2=0$. In the latter case, linear integrals (2.3) correspond directly to cyclic variables, as in [1].

According to what was stated in Section 2 system (3.6) may be reduced to a linearized system corresponding to (2.8). The characteristic equation (2.9) in this case has the form

$$a_0\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4=0$$

It is not difficult to work out expressions for the coefficients a_i in terms of the parameters of the system, but they will not be given here, as they are rather lengthy.

Earlier treatments [7] presented only the necessary conditions for the steady motions of the system to be stable and indicated the possibility of gyroscopic stabilization of some of them in the absence of dissipation.

According to the theorem proved in Section 2, a steady motion (3.1) is stable if the well-known Hurwitz conditions are satisfied

$$a_i>0,\quad i=1,\ldots,4;\quad a_1a_2a_3-a_1^2a_4-a_0a_3^2>0 \tag{3.7}$$

For rectilinear rolling of the system, conditions (3.7) are simplified and become *sufficient* conditions for stability

$$h_1>0,\quad \varepsilon=-1\quad (\varphi_{10}=\pi),\quad \omega^2>\omega_*^2,\quad \omega_*^2=\frac{(A+B_2-I)(mb-m_2d)}{C(C+mb^2-m_2d)},\quad m=m_1+m_2$$

These conditions mean that the centre of mass of the body M_2 (together with the rotor) lies below the centre of the wheel, and that the velocity of rolling is fairly high. If $\varepsilon=1\,(\varphi_{10}=0)$ or $\varepsilon=-1\,(\varphi_{10}=\pi)$ and $\omega^2<\omega_*^2$, then rectilinear rolling is unstable, since in these cases the coefficients a_3 and a_4 of the characteristic equation have different signs. In particular, if $\varphi_{10}=0$, rectilinear rolling is always unstable, as observed in [9] for a simpler monocycle model.

Now let the steady motion be spin around the vertical at a constant velocity Ω. Then $a_0 > 0, a_1 > 0$ for $h_1 > 0$.

We will now present asymptotic formulae (as $\Omega_r \to \infty$) for the coefficients a_2, a_3 and a_4, on the assumption that the spinning velocity of the rotor Ω_r is fairly high

$$a_2 = I^2\Omega_r^2 + \ldots > 0, \quad a_3 = Ih_1\varepsilon\Omega\Omega_r + \ldots, \quad a_4 = I^2\Omega^2\Omega_r^2 + \ldots > 0$$

In this case the sufficient conditions (3.7) for stable spinning become $\varepsilon\Omega\Omega_r > 0$, for any values of the geometrical and mass parameters of the system. This means that, in stable spinning, the directions of spin of the system at velocity Ω and of the spinning of the rotor at velocity Ω_r must be compatible with the position of the centre of mass of the body M_2 relative to the centre of the disk ($\varepsilon = \pm 1$).

In the case that the disk is weightless (more precisely, the mass of the disk is much less than that of the body M_2) and the body M_2 may be considered in calculations as a point mass, conditions (3.7) may be expressed as

$$(q-q_0)(q-q_1)>0, \quad qf_1>0, \quad q(q-q_0)f_2>0, \quad q(q-q_2)f_3>0 \tag{3.8}$$

where we have introduced the following dimensionless quantities

$$f_i = q^2 + G_iq + R_i, \quad i=1,2,3; \quad q = \varepsilon(I\Omega\Omega_r - m_2gd)/(m_2b^2\Omega^2)$$
$$q_0 = \delta^2, \qquad q_1 = \gamma - (1+\varepsilon\delta)^2, \qquad q_2 = -\varepsilon\delta\gamma$$
$$G_1 = -\gamma + (1+\varepsilon\delta), \qquad R_1 = -(1+\varepsilon\delta)\delta^2\gamma$$
$$G_2 = 2\varepsilon\delta\gamma + (1+\varepsilon\delta), \qquad R_2 = -\gamma\delta^2(1+\varepsilon\delta-\gamma)$$
$$G_3 = (1+\varepsilon\delta) - \gamma(1-\varepsilon\delta), \qquad R_3 = \varepsilon\delta\gamma[-\gamma + (1+\varepsilon\delta)(2+\varepsilon\delta)]$$
$$\delta = d/b, \qquad \gamma = g/(b\Omega^2)$$

Let $\delta = 1$ ($d = b$), $\varepsilon = +1$. Then the stability conditions (3.8) are satisfied in the following cases

$$\begin{aligned} &\text{if } I\Omega\Omega_r > 2m_2gb, && \text{for } \Omega^2 < g/b \\ &\text{if } I\Omega\Omega_r > m_2b(g+b\Omega^2), && \text{for } \Omega^2 > g/b \end{aligned} \tag{3.9}$$

Obviously conditions (3.9) can only be satisfied if the system and the rotor are spinning in the same sense.

Conditions (3.9) may be expressed in the following equivalent form

$$D \equiv I^2\Omega_r^2 - 4m_2^2b^3g > 0, \quad \Omega_1 < \Omega < \Omega_2 \tag{3.10}$$

where

$$\Omega_1 = 2m_2bg/(I\Omega_r), \quad \Omega_2 = (I\Omega_r + \sqrt{D})/(2m_2b^2)$$

Similar stability conditions may be derived for the case when $\delta = 1$ and $\varepsilon = -1$.

Remark. Conditions (3.10) are similar to the conditions for the stability of the steady motion of a gyroscope in gimbals, which is uniform rotations about the vertical of the outer ring at an angular velocity Ω and of the gyroscope at an angular velocity Ω_r [10, 11].

This research was partial by supported by the Russian Foundation for Basic Research (00-01-00391) and the "Universities of Russia" Programme (991736).

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Translated by D.L.